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ON RESULTS IN TWO COMPLETE FUZZY METRIC SPACES

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Abstract

In this paper, we have proved a new result concerning fixed point on two complete fuzzy metric spaces. This result generalizes well known fixed point theorems in complete metric spaces.

1. Introduction and Preliminaries

The fuzzy sets was introduced initially by Zadeh [17] in 1965. George and Veeramani [5] modified the concept of fuzzy metric space. Fisher [6], Aliouche and Fisher [1], Telci [15] proved some related fixed point theorems in compact metric spaces. Recently, Rao et.al [12] proved some related fixed point theorems in sequentially compact fuzzy metric spaces. Motivated by a work due to B. Fisher[6], we have observed that proving fixed point theorems using two complete fuzzy metric spaces is a generalization of fuzzy metric spaces with one contractive condition.

Key Words and Phrases : *Fuzzy metric space, Fixed Point, Cauchy sequences, Complete fuzzy metric spaces.*

AMS Subject Classification : 47H10, 54H25.

2. Preliminaries

Definition 2.1 : A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 2.2 : A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d,$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$

Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.3 : The $3-$tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t - norm$ and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t_1, t_2 > 0$

1. $M(x, y, 0) = 0.$
2. $M(x, y, t) = 1, t > 0$ if and only if $x = y.$
3. $M(x, y, t) = M(y, x, t)$
4. $M(x, z, t_1 + t_2) \geq M(x, y, t_1) * M(y, z, t_2)$
5. $M(x, y, .) : [0, \infty) \to [0,1]$ is left continuous.
6. $\lim_{t\to\infty} M(x, y, t) = 1.$

Example 2.1 : Let (X, d) be a metric space. Define $a * b = ab$ (or) $(a.b = min\{a, b\})$ and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y, z)}.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d as standard fuzzy metric.

Definition 2.4 : The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space in the sense of George and Veeramani if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

1. $M(x, y, t) > 0$

2. $M(x, y, t) = 1, t > 0$ if and only if $x = y$

3. $M(x, y, t) = M(y, x, t)$

4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$

5. $M(x, y, \cdot) : X^2 \times (0, \infty) \to [0, 1]$ is continuous.

Definition 2.5 : If (4) is replaced by condition $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$, then $(X, M, *)$ is called a strong fuzzy metric space.

Example 2.2 : Let $X = R$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Throughout the paper by $\varphi(x, y, t)$ is denote $\left(\frac{1}{M(x,y,t)} - 1\right)$ and $\phi(x, y, t)$ is denote $\left(\frac{1}{N(x,y,t)} - 1\right)$.

Definition 2.6 : Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if

$$\lim_{t\to\infty} \varphi(x_n, x_{n+p}, t) = 0 \text{ for all } t > 0 \text{ and } n, p \in N.$$

Definition 2.7 : Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be convergent to a point x $\in$ X if

$$\lim_{t\to\infty} \varphi(x_n, x, t) = 0 \text{ for all } t > 0.$$

Definition 2.8 : A fuzzy metric space X is said to be complete if every Cauchy sequence in X converges to some point in X.

Definition 2.9 : Let $(X, M, *)$ be a fuzzy metric space. We will say the mapping $T : X \to X$ is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\varphi(Tx, Ty, t) \leq k\varphi(x, y, t)$$

for each $x, y \in X$ and $t > 0$. (k is called the contractive constant of T.)

Lemma 2.1 : Let $\{x_n\}$ is a sequence in a fuzzy metric space X and if $\varphi(x_n, x_{n+1}, t) \leq k^n \varphi(x_0, x_1, t)$ where $0 < k < 1$, $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof : Suppose that $\varphi(x_n, x_{n+1}, t) \leq k^n \varphi(x_0, x_1, t)$ where $0 < k < 1$, and $t \geq 0$. Let m, n be two positive integers with $m \geq n$, say $m = n + p$, $p > 0$. Then we have

$$\begin{aligned}\varphi(x_n, x_{n+p}, t) &\leq \varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{n+2}, t) + \cdots + \varphi(x_{n+p-1}, x_{n+p}, t)\\ &\leq k^n \varphi(x_0, x_1, t) + k^{n+1}\varphi(x_0, x_1, t) + \cdots + k^{n+p-1}\varphi(x_0, x_1, t)\end{aligned}$$

Taking limit as $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} \varphi(x_n, x_{n+p}, t) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. □

The following theorem was proved by B. Fisher.

Theorem 2.1 : Let (X, d) and (Y, ρ) be complete metric spaces. If $T : X \to Y$ and $S : Y \to X$ satisfying the inequalities,

$$\rho(Tx, TSy) \leq c \max\left(d(x, Sy), \rho(y, Tx), \rho(y, TSy)\right)$$

$$d(Sy, STx) \leq c \max\left(\rho(y, Tx), d(x, Sy), d(x, STx)\right)$$

$\forall x \in X$ and $y \in Y$ where $0 \leq c < 1$ then ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further $Tz = w$ and $Sw = z$.

3. Main Results

Theorem 3.1 : Let $(X, M, *)$ and $(X, N, *)$ be a complete fuzzy metric spaces. if $T : X \to Y$ and $S : Y \to X$ satisfying the inequalities,

$$\begin{aligned}\phi^3(Tx, TSy, t) \leq k_1 \max[&\phi(y, Tx, t)\varphi(x, Sy, t)\phi(y, Tx, t),\\ &\phi(y, Tx, t)\phi(y, Tx, t)\phi(y, TSy, t),\\ &\phi(y, TSy, t)\varphi(x, Sy, t)\phi(y, TSy, t)]\end{aligned} \tag{1}$$

$$\begin{aligned}\varphi^3(Sy, STx, t) \leq k_2 \max[&\varphi(x, Sy, t)\phi(y, Tx, t)\varphi(x, Sy, t),\\ &\varphi(x, Sy, t)\varphi(x, Sy, t)\varphi(x, STx, t),\\ &\phi(y, Tx, t)\varphi(x, STx, t)\varphi(x, STx, t)]\end{aligned} \tag{2}$$

for all $x \in X$ and $y \in Y$ where $0 \leq k_1 k_2 < 1$ then ST has a unique fixed point $z \in X$ and TS has a unique fixed point $w \in Y$. Further $Tz = w$ and $Sw = z$.

Proof : Let x be an arbitrary point in X.

Define sequences $\{x_n\}$ and $\{y_n\} \in X$ and Y respectively by,

$$(ST)^n x = x_n, T(ST)^{n-1}x = y_n$$

for $n = 1, 2, ...$ Using inequality (2), we have,

$$\begin{aligned}
\varphi^3(x_n, x_{n+1}, t) &= \varphi(Sy_n, STx_n, t) \\
&\leq k_2 \max[\varphi(x_n, Sy_n, t)\phi(y_n, Sy_n, t)\varphi(x_n, Sy_n, t) \\
&\qquad \varphi(x_n, Sy_n, t)\varphi(x_n, Sy_n, t)\varphi(x_n, STx_n, t), \\
&\qquad \phi(y_n, Tx_n, t)\varphi(x_n, STx_n, t)\varphi(x_n, STx_n, t)] \\
\varphi^3(x_n, x_{n+1}, t) &\leq k_2 \max[\varphi(x_n, x_n, t)\phi(y_n, y_{n+1}, t)\varphi(x_n, x_n, t), \\
&\qquad \varphi(x_n, x_n, t)\varphi(x_n, x_n, t)\varphi(x_n, x_{n+1}, t) \\
&\qquad \phi(y_n, y_{n+1}, t)\varphi(x_n, x_{n+1}, t)\varphi(x_n, x_{n+1}, t)] \\
\varphi^3(x_n, x_{n+1}, t) &\leq k_2\phi(y_n, y_{n+1}, t)\varphi(x_n, x_{n+1}, t)\varphi(x_n, x_{n+1}, t)
\end{aligned}$$

Then

$$\varphi(x_n, x_{n+1}, t) \leq k_2\phi(y_n, y_{n+1}, t) \tag{3}$$

If $\varphi(x_n, x_{n+1}, t) \neq 0$ and by using inequality (1), we have

$$\begin{aligned}
\phi^3(y_n, y_{n+1}, t) &= \phi^3(Tx_{n-1}, TSy_n, t) \\
&\leq k_1 \max[\phi(y_n, Tx_{n-1}, t)\varphi(x_{n-1}, Sy_n, t)\phi(y_n, Tx_{n-1}, t), \\
&\qquad \phi(y_n, Tx_{n-1}, t)\phi(y_n, Tx_{n-1}, t)\phi(y_n, TSy_n, t), \\
&\qquad \phi(y_n, TSy_n, t)\varphi(x_{n-1}, Sy_n, t)\phi(y_n, TSy_n, t)] \\
\phi^3(y_n, y_{n+1}, t) &\leq k_1 \max[\phi(y_n, y_n, t)\varphi(x_{n-1}, x_n, t)\phi(y_n, y_n, t), \\
&\qquad \phi(y_n, y_n, t)\phi(y_n, y_n, t)\phi(y_n, y_{n+1}, t), \\
&\qquad \phi(y_n, y_{n+1}, t)\varphi(x_{n-1}, x_n, t)\phi(y_n, y_{n+1}, t)] \\
\phi^3(y_n, y_{n+1}, t) &\leq k_1\phi(y_n, y_{n+1}, t)\varphi(x_{n-1}, x_n, t)\phi(y_n, y_{n+1}, t)
\end{aligned}$$

Then

$$\phi(y_n, y_{n+1}, t) \leq k_1\varphi(x_{n-1}, x_n, t) \tag{4}$$

If $\phi(y_n, y_{n+1}, t) \neq 0$. It follows that,

$$\begin{aligned}
\varphi(x_n, x_{n+1}, t) &\leq k_2\phi(y_n, y_{n+1}, t) \\
&\leq k_1k_2\varphi(x_{n-1}, x_n, t) \leq ... \leq (k_1k_2)^n\varphi(x, x_1, t)
\end{aligned}$$

and since $0 \leq k_1 k_2 < 1$, $\{x_n\}$ is a Cauchy sequence with a limit $z \in X$ and $\{y_n\}$ is a Cauchy sequence with a limit $w \in Y$.
Now, by using inequalities (1), we have

$$\begin{aligned}
\phi^3(Tz, y_n, t) &= \phi^3(Tz, TSy_{n-1}, t) \\
&\leq k_1 \max[\phi(y_{n-1}, Tz, t)\varphi(z, Sy_{n-1}, t)\phi(y_{n-1}, Tz, t), \\
&\qquad \phi(y_{n-1}, Tz, t)\phi(y_{n-1}, Tz, t)\phi(y_{n-1}, TSy_{n-1}, t), \\
&\qquad \phi(y_{n-1}, TSy_{n-1}, t)\varphi(z, Sy_{n-1}, t)\phi(y_{n-1}, TSy_{n-1}, t)] \\
\phi^3(Tz, y_n, t) &\leq k_1 \max[\phi(y_{n-1}, Tz, t)\varphi(z, x_{n-1}, t)\phi(y_{n-1}, Tz, t), \\
&\qquad \phi(y_{n-1}, Tz, t)\phi(y_{n-1}, Tz, t)\phi(y_{n-1}, y_n, t), \\
&\qquad \phi(y_{n-1}, y_n, t)\varphi(z, x_{n-1}, t)\phi(y_{n-1}, y_n, t)]
\end{aligned}$$

Letting $n \to \infty$, we have $\phi^3(Tz, w, t) \neq 0$ and so $Tz = w$. Similarly we can prove that $Sw = z$ and so $STz = Sw = z, TSw = Tz = w$. Thus, ST has a fixed point z and TS has a fixed point w. Now suppose that ST has a second fixed point z_0. Then by using inequality (2), we have

$$\begin{aligned}
\varphi^3(z, z_0, t) &= \varphi^3(STz_0, STz, t) \\
&\leq k_2 \max[\varphi(z, STz_0, t)\phi(Tz_0, Tz, t)\varphi(z, STz_0, t), \\
&\qquad \varphi(z, STz_0, t)\varphi(z, STz_0, t)\varphi(z, STz, t), \\
&\qquad \phi(Tz_0, Tz, t)\varphi(z, STz, t)\varphi(z, STz, t)] \\
&= k_2 \phi(Tz_0, Tz, t)\varphi(z, z_0, t)\varphi(z, z_0, t)
\end{aligned}$$

Which implies, $\varphi(z, z_0, t) \leq k_2 \phi(Tz_0, Tz, t)$.
But by using inequality (1), we have,

$$\begin{aligned}
\phi^3(Tz, Tz_0, t) &= \phi(Tz_0, TSTz, t) \\
&\leq k_1 \max[\phi(Tz, Tz_0, t)\varphi(z_0, STz, t)\phi(Tz, Tz_0, t), \\
&\qquad \phi(Tz, Tz_0, t)\phi(Tz, TSTz, t)\phi(Tz, Tz_0, t), \\
&\qquad \phi(Tz, TSTz, t)\varphi(z, STz, t)\varphi(z, STz, t)] \\
&= k_1 \phi(Tz, Tz_0, t)\varphi(z_0, z, t)\phi(Tz, Tz_0, t)
\end{aligned}$$

Which implies, $\phi(Tz_0, Tz, t) \leq k_1 \varphi(z, z_0, t)$ and so,

$$\varphi(z, z_0, t) \leq k_2 \phi(Tz_0, Tz, t) \leq k_1 k_2 \varphi(z, z_0, t).$$

Since $0 \leq k_1k_2 < 1$, the uniqueness of z follows. Similarly w is the unique fixed point of TS.

If there exists $n \in N$ such that $\varphi(x_n, x_{n+1}, t) = 0$ or $\phi(y_n, y_{n+1}, t) = 0$. The theorem is evident. □

Example 3.1 : Let $(M, X, *), (N, Y, *)$ be two fuzzy metric spaces such that $\varphi(x, y, t) = \phi(x, y, t) = \frac{t}{t+|x-y|}$ and $X = [3, 5], Y = (0, 3)$.

Define $T : X \to Y$ and $S : Y \to X$ by

$$Tx = \begin{cases} 1, & \text{if } x \in [3, 4[, \\ 2, & \text{if } x \in [4, 5], \end{cases}$$

.

$$Sy = \begin{cases} 3, & \text{if } y \in (0, 1[, \\ 4, & \text{if } y \in [1, 3), \end{cases}$$

. We have $STx = 4$ for all $x \in [3, 5]$ and

$$TSy = \begin{cases} 1, & \text{if } y \in (0, 1[, \\ 2, & \text{if } y \in [1, 3), \end{cases}$$

. It is easy to see that, $ST(4) = 4, TS(2) = 2, T(4) = 2$ and $S(2) = 4$.

Corollary 3.1 : Let $(X, M, *)$ be a complete fuzzy metric space.

If $S, T : X \to X$ satisfying the inequalities,

$$\begin{aligned} \varphi^3(Tx, TSy, t) \leq k_1 \max[&\varphi(y, Tx, t)\varphi(x, Sy, t)\varphi(y, Tx, t), \\ &\varphi(y, Tx, t)\varphi(y, Tx, t)\varphi(y, TSy, t), \\ &\varphi(y, TSy, t)\varphi(x, Sy, t)\varphi(y, TSy, t)] \end{aligned}$$

$$\begin{aligned} \varphi^3(Sy, STx, t) \leq k_2 \max[&\varphi(x, Sy, t)\varphi(y, Tx, t)\varphi(x, Sy, t), \\ &\varphi(x, Sy, t)\varphi(x, Sy, t)\varphi(x, STx, t), \\ &\varphi(y, Tx, t)\varphi(x, STx, t)\varphi(x, STx, t)] \end{aligned}$$

for all $x, y \in X$ where $0 \leq k_1, k_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Proof : The existence of z and w follows from above Theorem. If $z = w$, then z is of course a common fixed point of S and T. Now suppose that T has a second fixed point

z_0. Then, by using inequality (2.5), we have,

$$\begin{aligned}\varphi^3(z,z_0,t) &= \varphi^3(Tz_0,TSz,t)\\ &\le k_1\max[\varphi(z,Tz_0,t)\varphi(z_0,Sz,t)\varphi(z,Tz_0,t),\\ &\qquad \varphi(z,Tz_0,t)\varphi(z,Tz_0,t)\varphi(z,TSz,t),\\ &\qquad \varphi(z,TSz,t)\varphi(z_0,Sz,t)\varphi(z,TSz,t)]\\ &= k_1\varphi^3(z_0,z,t)\end{aligned}$$

Since $0 \le k_1 < 1$, the uniqueness of z follows. Similarly z is the unique fixed point of S. □

Corollary 3.2 : Let $(X, M, *)$ be a complete fuzzy metric space. If $S, T : X \to X$ satisfying the inequalities,

$$\begin{aligned}\varphi^3(Tx,TSy,t) &\le [k_1\varphi(y,Tx,t)\varphi(x,Sy,t)\varphi(y,Tx,t)\\ &\quad + l_1\varphi(y,Tx,t)\varphi(y,Tx,t)\varphi(y,TSy,t)\\ &\quad + m_1\varphi(y,TSy,t)\varphi(x,Sy,t)\varphi(y,TSy,t)]\\ \varphi^3(Sy,STx,t) &\le [k_2\varphi(x,Sy,t)\varphi(y,Tx,t)\varphi(x,Sy,t)\\ &\quad + l_2\varphi(x,Sy,t)\varphi(x,Sy,t)\varphi(x,STx,t)\\ &\quad + m_2\varphi(y,Tx,t)\varphi(x,STx,t)\varphi(x,STx,t)]\end{aligned}$$

where $\frac{k_1+l_1}{1-m_1} < 1$ and $\frac{k_2+l_2}{1-m_2} < 1$, for all $x, y \in X$ and $k_1, l_1, m_1, k_2, l_2, m_2 > 0$. Then ST has a unique fixed point z and TS has a unique fixed point w. Further $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Proof : The proof of corollary follows immediate. □

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